

# On $L_n$ -Injective Modules and $L_n$ -Injective Dimensions

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## 1. INTRODUCTION

Throughout this paper,  $R$  is an associative ring with identity and all modules are left  $R$ -modules unless otherwise stated. We also use  ${}_R\mathfrak{M}$  to denote the category of left  $R$ -modules,  $w.\text{gl.dim}(R)$  (resp.  $\text{gl.dim}(R)$ ) to denote the weak global (resp. global) dimension of  $R$ ,  $\mathcal{F}_n$  to denote the class of all  $R$ -modules with flat dimension at most  $n$ . For an  $R$ -module  $M$ ,  $\text{pd}_R M$  (resp.  $\text{fd}_R M$ ) stands for the projective (resp. flat) dimension of  $M$ ,  $\text{id}_R M$  (resp.  $\text{cd}_R M$ ) stands for the injective (resp. cotorsion) dimension of  $M$ .

Cotorsion modules have been received a lot of attention in many articles, see [2, 4, 21]. As in [4], an  $R$ -module  $C$  is called cotorsion if  $\text{Ext}_R^1(F, C) = 0$  for all flat module  $F$ . A celebrated result that was proved by Bican et al. in [2] is the Flat Cover Conjecture (FCC): Over any ring, every module has a flat cover and hence every module has a cotorsion envelope.

On the further development on the idea of cotorsion module notion, in [13], the weak-injective modules have been studied by Lee. Recall from [13] that an  $R$ -module  $W$  is called weak-injective if  $\text{Ext}_R^1(M, W) = 0$  for all modules  $M$  with  $\text{fd}_R M \leq 1$  and from [3] that a domain  $R$  is called almost perfect (APD shortly) if all its proper homomorphic image are perfect. It was proved in [8, Corollary 6.4.8] that a domain  $R$  is an APD if and only if every module of flat dimension  $\leq 1$  has projective dimension  $\leq 1$ ; if and only if every divisible module is weak-injective; if and only if every epic image of a weak-injective module is weak-injective.

In 2012 the notion of  $n$ -cotorsion modules was introduced in [6] by Enochs and Huang. An  $R$ -module  $N$  is called  $n$ -cotorsion in [6] if  $\text{Ext}_R^1(M, N) = 0$  for all  $R$ -module  $M$  with  $\text{fd}_R M \leq n$ . But the name of the  $n$ -cotorsion module has been used by Mao and Ding. In [15] the  $n$ -cotorsion module  $N$  means  $\text{Ext}_R^{n+1}(F, N) = 0$  for any flat  $R$ -module  $F$ . The two notions of  $n$ -cotorsion modules are not coincident, see Example 2.2.

In this paper, the  $n$ -cotorsion modules which are defined in [6], following Lee's idea, are said to be  $L_n$ -injective modules. Thus weak-injective modules are exactly  $L_1$ -injective modules. Denote  $\mathcal{F}_n$  and  $\mathcal{L}_n$  the classes of modules of flat dimension  $\leq n$  and of  $L_n$ -injective modules, respectively.

In Section 2, we prove in Corollary 2.7 that the  $n$ -th cosyzygy of a cotorsion  $R$ -module  $M$  is  $L_n$ -injective, and in Theorem 2.8 that an  $R$ -module  $M$  has flat dimension  $\leq n$  if and only if  $\text{Ext}_R^i(M, N) = 0$  for any  $L_n$ -injective module  $N$  and any  $i \geq 1$ ,

Given two classes  $\mathcal{A}$  and  $\mathcal{B}$  of  $R$ -modules, set  $\mathcal{A}^\perp = \{B \mid \text{Ext}_R^1(A, B) = 0 \text{ for all } A \in \mathcal{A}\}$  and  ${}^\perp\mathcal{B} = \{A \mid \text{Ext}_R^1(A, B) = 0 \text{ for all } B \in \mathcal{B}\}$ , which are called the right orthogonal class of  $\mathcal{A}$  and the left orthogonal class of  $\mathcal{B}$ , respectively. A pair  $(\mathcal{A}, \mathcal{B})$  of  $R$ -modules is called a cotorsion theory (or cotorsion pair) [5] if  $\mathcal{A}^\perp = \mathcal{B}$  and  $\mathcal{A} = {}^\perp\mathcal{B}$ . A cotorsion theory  $(\mathcal{A}, \mathcal{B})$  is said to be complete [19] if every  $R$ -module has a special  $\mathcal{A}$ -precover. It is shown in [2] that the pair  $(\mathcal{F}, \mathcal{C})$  is a complete cotorsion theory, where  $\mathcal{F}$  and  $\mathcal{C}$  are the classes of flat  $R$ -modules and cotorsion modules, respectively. In 2006, it is shown in [13] that the pair  $(\mathcal{F}_1, \mathcal{W})$  is a cotorsion theory when  $R$  is a domains, where  $\mathcal{F}_1$  and  $\mathcal{W}$  denote the classes of  $R$ -modules with flat dimension at most 1 and weak-injective modules, respectively. In Section 3, for the further examination, in Theorem 3.7, we prove that the pair  $(\mathcal{F}_n, \mathcal{L}_n)$  is a complete cotorsion theory, where  $\mathcal{F}_n$  and  $\mathcal{L}_n$  are the classes of  $R$ -modules with flat dimension at most  $n$  and  $L_n$ -injective modules, respectively.

In Sections 4 and 5, we are going to introduce the  $\mathcal{L}_n$ -injective dimension of modules and the  $\mathcal{L}_n$ -global dimension of rings. In Section 6, we start by discussing when  $L_n$ -injective modules are injective. It is shown in Theorem 6.1 that  $L_n$ -injective modules are injective if and only if  $w.\text{gl.dim}(R) \leq n$ . Then we discuss, for  $n \geq 1$ , when every  $R$ -module is  $L_n$ -injective.

It was shown in [21] that every module is cotorsion if and only if  $R$  is left perfect. In Theorem 6.5, we show that every module is  $L_n$ -injective if and only if  $R$  is left perfect with  $l.\text{FFD}(R) = 0$ , where  $l.\text{FFD}(R)$  is the left finitistic weak dimension of  $R$  defined in [1]. We introduce also the notion of  $L_n$ -hereditary rings and give a series of consideration on them. Compared the notions of almost perfect rings and  $L_1$ -hereditary rings, we point out that all almost perfect rings are  $L_1$ -hereditary and a domain  $R$  is an APD if and only if  $R$  is  $L_1$ -hereditary, and in Example 7.4, we give an example that some  $L_1$ -hereditary ring is not perfect.

## 2. $L_n$ -INJECTIVE MODULES

We start this section with the following difinition.

**Definition 2.1.** (see [6, Definition 2.7(1)]) *An  $R$ -module  $W$  is called  $L_n$ -injective ( $L$  means Lee) if  $\text{Ext}_R^1(M, W) = 0$  for all  $R$ -modules  $M \in \mathcal{F}_n$ .*

Naturally,  $L_0$ -injective modules are exactly cotorsion modules and  $L_1$ -injective modules are exactly weak-injective modules.

We denote the class of all  $L_n$ -injective  $R$ -modules by  $\mathcal{L}_n$ .

**Example 2.2.** *It is easy to see that  $L_n$ -injective modules are  $n$ -cotorsion modules under the definition of Mao et al. But  $n$ -cotorsion modules are not necessarily  $L_n$ -injective modules. For example, take  $n = 1$  and let  $R = \mathbb{Z}$  and  $M = R/(2)$ . Then  $R$  is a 1-cotorsion module since  $\text{id}_R R = 1$ . It*

is clear that  $\text{fd}_R M = 1$  and  $\text{Ext}_R^1(M, R) \cong R/2R \neq 0$  by [17, Theorem 7.17]. Hence then  $R$  is not  $L_1$ -injective.

**Example 2.3.** The following facts are true.

- (1) Injective modules are  $L_n$ -injective modules for all integer  $n \geq 0$ .
- (2) If  $m \leq n$ , then  $L_n$ -injective  $R$ -modules are  $L_m$ -injective, and hence  $\mathcal{L}_m \supseteq \mathcal{L}_n$ .
- (3) Let  $\{W_i\}$  be a family of  $R$ -modules. Then  $\prod_i W_i$  is  $L_n$ -injective if and only if each  $W_i$  is  $L_n$ -injective modules.

- (4) Let  $n \geq 1$  and let  $R$  be a domain. Suppose  $W$  is an  $L_n$ -injective module. For  $a \in R$  and  $a \neq 0$ , since  $\text{Ext}_R^1(R/Ra, W) = 0$ , we have that  $W$  is divisible.

**Proposition 2.4.** The following statements are equivalent for an  $R$ -module  $W$ :

- (1)  $W$  is  $L_n$ -injective.
- (2) For any  $R$ -module  $M \in \mathcal{F}_n$  and any integer  $k \geq 1$ , then  $\text{Ext}_R^k(M, W) = 0$ .
- (3) Any exact sequence  $0 \rightarrow W \rightarrow B \rightarrow C \rightarrow 0$  with  $C \in \mathcal{F}_n$  is split.
- (4) For any exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  with  $C \in \mathcal{F}_n$ , then the sequence  $0 \rightarrow \text{Hom}(C, W) \rightarrow \text{Hom}(B, W) \rightarrow \text{Hom}(A, W) \rightarrow 0$  is exact.

**Proof.** (1)  $\Leftrightarrow$  (3)  $\Leftrightarrow$  (4) and (2)  $\Rightarrow$  (1) are clear.

(1)  $\Rightarrow$  (2). From the definition we have  $\text{Ext}_R^1(M, W) = 0$ , that is, the assertion is true for the case  $k = 1$ .

Assume  $k > 1$ . Let  $0 \rightarrow A \rightarrow F \rightarrow M \rightarrow 0$  be exact, where  $F$  is projective. Thus  $\text{Ext}_R^k(M, W) \cong \text{Ext}_R^{k-1}(A, W)$ . Note that  $A \in \mathcal{F}_n$ . Hence  $\text{Ext}_R^k(M, W) = 0$  by induction on  $k$ .  $\square$

**Proposition 2.5.** Let  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  be an exact sequence. Then:

- (1) If  $A \in \mathcal{L}_n$ , then  $B \in \mathcal{L}_n$  if and only if  $C \in \mathcal{L}_n$ .
- (2) If  $C \in \mathcal{F}_n$ , then  $A \in \mathcal{F}_n$  if and only if  $B \in \mathcal{F}_n$ .

**Proof.** These are straightforward.  $\square$

Let  $\mathcal{C}$  be a class of  $R$ -modules and  $M$  a  $R$ -module. For a  $\mathcal{C}$ -resolution of  $M$

$$\cdots \rightarrow C_n \rightarrow \cdots \rightarrow C_1 \rightarrow C_0 \rightarrow M \rightarrow 0, \quad (\text{resp. } 0 \rightarrow M \rightarrow C^0 \rightarrow C^1 \rightarrow \cdots),$$

set  $K_0 = M$ ,  $K_1 = \ker(C_0 \rightarrow M)$ ,  $K_i = \ker(C_{i-1} \rightarrow C_{i-2})$  (resp.  $Q^0 = M$ ,  $Q^1 = \text{cok}(M \rightarrow C^0)$ ,  $Q^i = \text{cok}(C_{i-2} \rightarrow C_{i-1})$ ) for  $i \geq 2$ . The  $n$ -th kernel  $K_n$  (resp. cokernel  $Q^n$ ) ( $n \geq 0$ ) is called the  $n$ -th  $\mathcal{C}$ -syzygy (resp.  $\mathcal{C}$ -cosyzygy) of  $M$ . In particular, if  $\mathcal{C}$  is the class of projective modules (resp. flat modules), then  $K_n$  is simply called the  $n$ -th syzygy (resp.  $n$ -york) of  $M$ ; and if  $\mathcal{C}$  is the class of injective modules, then  $Q^n$  is simply called the  $n$ -th cosyzygy of  $M$ .

**Theorem 2.6.** Let  $n$  and  $m$  be two given nonnegative integers and let  $W$  be an  $L_n$ -injective module. Then the  $m$ -th cosyzygy of  $W$  is  $L_{n+m}$ -injective.

**Proof.** The case  $m = 0$  is clear. Now assume  $m > 0$ . Let  $W'$  be an  $m$ -th cosyzygy of  $W$  and let  $M \in \mathcal{F}_{n+m}$ . Let  $B$  be an  $(m-1)$ -th syzygy of  $M$ . Then  $\text{fd}_R B \leq n$ . Hence  $\text{Ext}_R^1(M, W') \cong \text{Ext}_R^{m+1}(M, W) \cong \text{Ext}_R^1(B, W) = 0$ . Therefore,  $W' \in \mathcal{L}_{n+m}$ .  $\square$

**Corollary 2.7.** Let  $C$  be a cotorsion module. then the  $n$ -th cosyzygy of  $C$  is  $L_n$ -injective.

Corresponding with the consideration of flat dimension at most  $n$ , Fuchs and Lee have proved that if  $R$  is an integral domain, then an  $R$ -module  $M$  satisfies  $\text{fd}_R M \leq n$  if and only if  $\text{Ext}_R^n(M, W) = 0$  for any  $L_1$ -injective module  $W$  (See [14, Theorem 2.2] and [9, Lemma 5.5]). The following is the further discussion on the modules with flat dimension  $\leq n$ .

**Theorem 2.8.** *Let  $R$  be any ring and  $M$  be an  $R$ -module and  $n \geq 1$ . Then the following are equivalent:*

- (1)  $\text{fd}_R M \leq n$ .
- (2)  $\text{Ext}_R^1(M, N) = 0$  for any  $L_n$ -injective module  $N$ .
- (3)  $\text{Ext}_R^i(M, N) = 0$  for any  $L_n$ -injective module  $N$  and any  $i \geq 1$ .
- (4) If  $0 \leq m \leq n$ , then  $\text{Ext}_R^{n-m+1}(M, N) = 0$  for any  $L_m$ -injective module  $N$ .
- (5) If  $0 \leq m \leq n$ , then  $\text{Ext}_R^{n-m+i}(M, N) = 0$  for any  $L_m$ -injective module  $N$  and any  $i \geq 1$ .
- (6)  $\text{Ext}_R^n(M, W) = 0$  for any weak-injective ( $L_1$ -injective) module  $W$ .
- (7)  $\text{Ext}_R^i(M, W) = 0$  for any weak-injective module  $W$  and any  $i \geq n$ .
- (8)  $\text{Ext}_R^{n+1}(M, C) = 0$  for any cotorsion module  $C$ .
- (9)  $\text{Ext}_R^{n+i}(M, C) = 0$  for any cotorsion module  $C$  and any  $i \geq 1$ .

**Proof.** (3)  $\Rightarrow$  (2) and (9)  $\Rightarrow$  (8) and (7)  $\Rightarrow$  (6) are clear.

(1)  $\Rightarrow$  (3). It follows by Proposition 2.4.

(5)  $\Rightarrow$  (9). It follows by picking  $m = 0$ .

(5)  $\Rightarrow$  (7). It follows by picking  $m = 1$ .

(4)  $\Rightarrow$  (6). It follows by picking  $m = 1$ .

(5)  $\Rightarrow$  (3). It follows by picking  $m = n$ .

(2)  $\Rightarrow$  (4). Let  $0 \rightarrow N \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_{n-m-1} \rightarrow W \rightarrow 0$  be exact, where  $E_0, E_1, \dots, E_{n-m-1}$  be injective. Thus  $W$  is an  $(n-m)$ -cosyzygy of  $N$ . By Theorem 2.6,  $W$  is  $L_n$ -injective. Thus  $\text{Ext}_R^{n-m+1}(M, N) \cong \text{Ext}_R^1(M, W) = 0$ .

(3)  $\Rightarrow$  (5). It is similar to the proof of (2)  $\Rightarrow$  (4).

(6)  $\Rightarrow$  (8). Let  $0 \rightarrow C \rightarrow E \rightarrow W \rightarrow 0$  be exact, where  $E$  is injective. By Corollary 2.7,  $W$  is weak-injective. Hence  $\text{Ext}_R^{n+1}(M, C) \cong \text{Ext}_R^n(M, W) = 0$ .

(8)  $\Rightarrow$  (1). Let  $K$  be an  $n$ -syzygy of  $M$ . Then, for any cotorsion module  $C$ ,  $\text{Ext}_R^1(K, C) \cong \text{Ext}_R^{n+1}(M, C) = 0$ . Hence  $K$  is flat by [21, Lemma 3.4.1]. Therefore,  $\text{fd}_R M \leq n$ .  $\square$

Let  $R$  be a ring. Bass defined in [1] the left weak finitistic dimension of  $R$  as follow:

$$l.\text{FFD}(R) = \sup\{\text{fd}_R M \mid M \text{ is an } R\text{-module with } \text{fd}_R M < \infty\}.$$

**Theorem 2.9.** *Let  $n < m$  be two given integers. Then the following statements are equivalent for a ring  $R$ :*

- (1)  $l.\text{FFD}(R) \leq n$ .
- (2)  $\mathcal{F}_m = \mathcal{F}_n$ .
- (3)  $\mathcal{L}_m = \mathcal{L}_n$ .
- (4) Every  $L_n$ -injective module is  $L_m$ -injective.

**Proof.** (1)  $\Rightarrow$  (2). It is clear.

(2)  $\Rightarrow$  (3). It follows from the facts  $\mathcal{L}_m = \mathcal{F}_m^\perp$  and  $\mathcal{L}_n = \mathcal{F}_n^\perp$ .

(3)  $\Rightarrow$  (4). It is trivial.

(4)  $\Rightarrow$  (1). It is enough that we assume  $m = n + 1$ . Let  $M$  be an  $R$ -module with  $\text{fd}_R M = s < \infty$ . If  $s > n$ , without loss of generality, we can assume  $s = n + 1$ . Hence we have  $\text{Ext}_R^1(M, W) = 0$  for any  $L_n$ -injective module  $W$  by hypothesis. And so  $\text{fd}_R M \leq n$  by Theorem 2.8, a contradiction. Therefore,  $\text{fd}_R M \leq n$ , and hence  $l.\text{FFD}(R) \leq n$ .  $\square$

**Corollary 2.10.** *Let  $n \geq 1$ . Then every cotorsion  $R$ -module is  $L_n$ -injective if and only if  $l.\text{FFD}(R) = 0$ .*

**Proof.** Pick  $m = 0$  in Theorem 2.9.  $\square$

### 3. THE COTORSION THEORY $(\mathcal{F}_n, \mathcal{L}_n)$

The goal of this section is to show  $(\mathcal{F}_n, \mathcal{L}_n)$  is a complete cotorsion theory for an arbitrary ring  $R$ .

**Definition 3.1.** *A right  $R$ -module  $D$  is called  $L_n$ -flat ( $L$  means Lee) if  $\text{Tor}_1^R(D, M) = 0$  for all left  $R$ -modules  $M \in \mathcal{F}_n$ .*

Clearly, flat modules are  $L_n$ -flat. In [6]  $L_n$ -flat modules are called  $n$ -torsionfree. In the following we denote  $\mathcal{D}_n$  the class of  $L_n$ -flat modules.

For a right  $R$ -module  $D$ , write  $D^+ = \text{Hom}_Z(D, \mathbb{Q}/\mathbb{Z})$ , which is called the character module of  $D$ .

**Lemma 3.2.** (1)  $M^+$  is pure injective for every  $R$ -module  $M$ .

(2) An  $R$ -module  $M$  is flat if and only if  $M^+$  is injective.

(3) Every pure injective  $R$ -module is cotorsion.

(4)  $M^+$  is cotorsion for every  $R$ -module  $M$ .

**Proof.** (1) and (2) by [13, Lemma 2.2].

(3) See [5, Lemma 5.3.23].

(4) It follows by (1) and (3).  $\square$

**Proposition 3.3.** *A right  $R$ -module  $D$  is  $L_n$ -flat if and only if  $D^+$  is  $L_n$ -injective.*

**Proof.** It follows by the following standard isomorphism

$$(\text{Tor}_1^R(D, M))^+ \cong \text{Ext}_R^1(M, D^+),$$

where  $M \in \mathcal{F}_n$ .  $\square$

**Lemma 3.4.** *Let  $X$  be a right  $R$ -module. Then any  $n$ -york of  $X$  is  $L_n$ -flat.*

**Proof.** Let  $0 \rightarrow D \rightarrow D_{n-1} \rightarrow \cdots \rightarrow D_1 \rightarrow D_0 \rightarrow X \rightarrow 0$  be exact, where  $D_0, D_1, \dots, D_{n-1}$  are flat. Thus  $(D_0)^+, (D_1)^+, \dots, (D_{n-1})^+$  are injective and  $X^+$  is cotorsion by lemma 3.2 and  $0 \rightarrow X^+ \rightarrow (D_0)^+ \rightarrow (D_1)^+ \rightarrow \cdots \rightarrow (D_{n-1})^+ \rightarrow D^+ \rightarrow 0$  is exact. By Corollary 2.7,  $D^+$  is  $L_n$ -injective. By Proposition 3.3,  $D$  is  $L_n$ -flat.

**Theorem 3.5.** *Let  $M$  be an  $R$ -module. Then  $\text{fd}_R M \leq n$  if and only if  $\text{Tor}_1^R(D, M) = 0$  for any  $D \in \mathcal{D}_n$ .*

**Proof.** Suppose  $\text{fd}_R M \leq n$ . Then it is clear that  $\text{Tor}_1^R(D, M) = 0$  for any  $D \in \mathcal{D}_n$ . For the converse, let  $X$  be a right  $R$ -module and let  $D$  be the  $n$ -th yolk of  $X$ . Then  $D$  is  $L_n$ -flat by Lemma 3.4. Therefore,  $\text{Tor}_{n+1}^R(X, M) \cong \text{Tor}_1^R(Y, M) = 0$  by hypothesis. Hence  $\text{fd}_R M \leq n$ .  $\square$

Let  $(\mathcal{A}, \mathcal{B})$  be a cotorsion theory. Recall that  $(\mathcal{A}, \mathcal{B})$  is called hereditary if whenever  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is an exact sequence with  $B, C \in \mathcal{A}$ , then  $A$  is also in  $\mathcal{A}$ ; equivalently, whenever  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  is an exact sequence with  $A, B \in \mathcal{B}$ , then  $C$  is also in  $\mathcal{B}$ . Recall that  $(\mathcal{A}, \mathcal{B})$  is said to be complete [19] if every  $R$ -module has a special  $\mathcal{A}$ -precover. By [19, Lemma 1.13], a cotorsion theory  $(\mathcal{A}, \mathcal{B})$  is complete if and only if every  $R$ -module has a special  $\mathcal{B}$ -preenvelope.

Let  $\mathcal{A}$  be a class of left  $R$ -modules and let  $\mathcal{B}$  be a class of right  $R$ -modules. Write

$${}^\top \mathcal{A} = \{D \in \mathfrak{M}_R \mid \text{Tor}_1^R(D, M) = 0 \text{ for any } M \in \mathcal{A}\},$$

and

$$\mathcal{B}^\top = \{Y \in {}_R\mathfrak{M} \mid \text{Tor}_1^R(D, Y) = 0 \text{ for any } D \in \mathcal{B}\}.$$

If  $\mathcal{B} = {}^\top \mathcal{A}$  and  $\mathcal{A} = \mathcal{B}^\top$ , then the pair  $(\mathcal{A}, \mathcal{B})$  is called a Tor-torsion theory.

For a class  $\mathcal{C}$  of modules, set

$$\mathfrak{S}_{\mathcal{C}} = ({}^\perp \mathcal{C}, ({}^\perp \mathcal{C})^\perp).$$

**Lemma 3.6.** (1) If  $(\mathcal{A}, \mathcal{B})$  be a Tor-torsion theory, then  $\mathfrak{C} := (\mathcal{A}, \mathcal{A}^\perp)$  is a cotorsion theory. Moreover, if we write  $\mathcal{C} = \{B^+ \mid B \in \mathcal{B}\}$ , then  $\mathcal{C}$  is a subclass of the class of pure-injective modules and  $\mathfrak{S}_{\mathcal{C}} = \mathfrak{C}$ .

(2) Let  $\mathfrak{S}_{\mathcal{C}} = (\mathcal{A}, \mathcal{B})$  be a cotorsion theory. If  $\mathcal{C}$  is a subclass of the class of pure-injective modules, the  $\mathfrak{S}_{\mathcal{C}}$  is a complete cotorsion theory. Moreover, every  $R$ -module has an  $\mathcal{A}$ -cover and a  $\mathcal{B}$ -envelope.

**Proof.** See [19, Lemma 1.11& Theorem 2.8].  $\square$

**Theorem 3.7.** (1)  $(\mathcal{F}_n, \mathcal{D}_n)$  is a Tor-torsion theory.

(2)  $(\mathcal{F}_n, \mathcal{L}_n)$  is a complete cotorsion theory.

(3) Every  $R$ -module has a special  $\mathcal{F}_n$ -cover and a special  $\mathcal{L}_n$ -envelope. Further, for any  $R$ -module  $M$  and  $N$ , there are exact sequences

$$0 \rightarrow A \rightarrow F \rightarrow M \rightarrow 0 \quad \text{and} \quad 0 \rightarrow N \rightarrow W \rightarrow B \rightarrow 0, \quad (3.1)$$

where  $F$  is the  $\mathcal{F}_n$ -cover,  $W$  is the  $\mathcal{L}_n$ -envelope,  $B \in \mathcal{F}_n$ , and  $A \in \mathcal{L}_n$ .

**Proof.** (1) Clearly,  $\mathcal{D}_n = {}^\top \mathcal{F}_n$ . By Theorem 3.5,  $\mathcal{F}_n = \mathcal{D}_n^\top$ .

(2) Clearly,  $\mathcal{L}_n = \mathcal{F}_n^\perp$ . Hence  $(\mathcal{F}_n, \mathcal{L}_n)$  is a cotorsion theory by Lemma 3.6 (1). Moreover,  $(\mathcal{F}_n, \mathcal{L}_n)$  is a complete cotorsion theory by Lemma 3.6 (2).

(3) By Lemma 3.6 (2),  $M$  has the  $\mathcal{F}_n$ -cover  $\phi : F \rightarrow M$  and  $N$  has the  $\mathcal{L}_n$ -envelope  $\varphi : N \rightarrow W$ . Since  $M$  has a special  $\mathcal{F}_n$ -precover,  $\phi$  is epic. Because  $\mathcal{F}_n$  is closed under extension,  $A := \ker(\phi) \in \mathcal{L}_n$  by [21, Lemma 2.1.1]. Hence  $\phi : F \rightarrow M$  is also special.

The other statement is dual to the argument above, but we need to apply [21, Lemma 2.1.2].  $\square$

In the following we denote  $F_n(M)$  the  $\mathcal{F}_n$ -cover of an  $R$ -module  $M$  and  $L_n(N)$  the  $\mathcal{L}_n$ -envelope of an  $R$ -module  $N$ .

**Theorem 3.8.** *Let  $M$  and  $N$  be  $R$ -modules. Then the following statements are true:*

- (1)  *$M$  is  $L_n$ -injective if and only if  $F_n(M)$  is  $L_n$ -injective.*
- (2)  *$\text{fd}_R N \leq n$  if and only if  $\text{fd}_R L_n(N) \leq n$ .*

**Proof.** It follows directly from the exact sequences in (3.1) □

**Corollary 3.9.** *Let  $M$  and  $N$  be  $R$ -modules. Then the following statements are true:*

- (1)  *$M$  is cotorsion if and only if so is its flat cover.*
- (2)  *$N$  is flat if and only if so is its cotorsion envelope.*

**Theorem 3.10.** *Let  $N \subseteq C$  be an extension of  $R$ -modules and  $C$  be  $L_n$ -injective. Then the following statements are equivalent:*

- (1)  *$C$  is an  $\mathcal{L}_n$ -envelope of  $N$ .*
- (2)  *$C/N \in \mathcal{F}_n$ , and there is no submodule  $0 \neq A \subseteq C$  such that  $N \cap A = 0$  and  $C/(N + A) \in \mathcal{F}_n$ .*

**Proof.** (1) $\Rightarrow$ (2). Let  $C$  be an  $\mathcal{L}_n$ -envelope of  $N$  and  $i : N \rightarrow C$  be an inclusion homomorphism.  $C/N \in \mathcal{F}_n$  holds by Theorem 3.7. Now, let  $A$  be a submodule of  $C$  such that  $A \cap N = 0$  and  $B := C/(N + A) \in \mathcal{F}_n$ . Then the natural homomorphism  $\phi : N \rightarrow C/A$  is monic and  $\text{cok}(\phi) = C/(N + A)$ . Let  $L$  be an  $\mathcal{L}_n$ -envelope of  $C/A$ . Then we get the following commutative diagram with exact rows

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & N & \xrightarrow{\phi} & C/A & \longrightarrow & B \longrightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \longrightarrow & N & \xrightarrow{\phi} & L & \longrightarrow & X \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & Y & \xlongequal{\quad} & Y \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

$X \in \mathcal{F}_n$  since  $Y, B \in \mathcal{F}_n$ . Let  $\pi : C \rightarrow C/A$  be a natural homomorphism and  $\lambda : C/A \rightarrow L$  an inclusion homomorphism. Let  $f : L \rightarrow C$  be a homomorphism such that  $f\phi = i$ . Then  $f\lambda\pi$  is an isomorphism and  $\pi$  is monic. Hence  $A = 0$ .

(2) $\Rightarrow$ (1). Let  $E$  be an  $\mathcal{L}_n$ -envelope of  $N$  and  $\lambda : N \rightarrow E$  an inclusion homomorphism. Then  $D := E/N \in \mathcal{F}_n$ . So there exist homomorphisms  $f : E \rightarrow C, g : C \rightarrow E$  such that  $f\lambda = i, gi = \lambda$ . Thus  $(gf)\lambda = \lambda$ , where  $gf : E \rightarrow E$ . So  $fg$  is an isomorphism and there exist homomorphism  $h : E \rightarrow E$  such that  $hgf = \mathbf{1}_E$ . Thus  $C = \text{Im}(f) \oplus \ker(hg)$ . Set  $A = \ker(hg)$ . Then  $N \cap A = 0$  and  $C/(N + A) \cong E/N \in \mathcal{F}_n$ . By hypothesis,  $A = 0$ . Hence  $f : E \rightarrow C$  is an isomorphism. □

**Theorem 3.11.** *Let  $n < m$  be two given integers. Then the following statements are equivalent for a ring  $R$ :*

- (1)  *$l.\text{FFD}(R) \leq n$ .*
- (2)  *$\mathcal{D}_m = \mathcal{D}_n$ .*
- (3) *Every  $L_n$ -flat module is  $L_m$ -flat.*

**Proof.** (1) $\Rightarrow$ (2). By Theorem 2.9,  $\mathcal{F}_m = \mathcal{F}_n$ . Hence, by Theorem 3.7,

$$\mathcal{D}_m = {}^\top \mathcal{F}_m = {}^\top \mathcal{F}_n = \mathcal{D}_n.$$

(2) $\Rightarrow$ (1).  $\mathcal{F}_m = \mathcal{D}_m^\top = \mathcal{D}_n^\top = \mathcal{F}_n$  by applying Theorem 3.7 again.

(2) $\Leftrightarrow$ (3). Clearly. □

#### 4. $L_n$ -INJECTIVE DIMENSIONS OF MODULES

In this section, we study the  $L_n$ -injective modules over an arbitrary ring  $R$ .

Let  $M$  be an  $R$ -module. If there exists an exact sequence

$$0 \rightarrow M \rightarrow W_0 \rightarrow W_1 \rightarrow \cdots \rightarrow W_{m-1} \rightarrow W_m \rightarrow \cdots \quad (4.1)$$

in which each  $W_i$  is  $L_n$ -injective, then this exact sequence is called an  $L_n$ -injective resolution of  $M$ . Certainly, every  $R$ -module  $M$  has an  $L_n$ -injective resolution. If the following homomorphisms

$$M \rightarrow W_0, \quad \text{cok}(M \rightarrow W_0) \rightarrow W_1, \quad \cdots, \quad \text{cok}(W_{i-2} \rightarrow W_{i-1}) \rightarrow W_i, \cdots \quad (4.2)$$

are  $\mathcal{L}_n$ -envelopes, then the sequence (4.1) is called a minimal  $L_n$ -injective resolution of  $M$ .

**Proposition 4.1.** *Every  $R$ -module  $M$  has a minimal  $L_n$ -injective resolution.*

**Proof.** By Theorem 3.7,  $M$  has an  $\mathcal{L}_n$ -envelope  $W_0$ . Note  $C_0 = \text{cok}(M \rightarrow W_0)$ . Then  $C_0$  also has an  $\mathcal{L}_n$ -envelope  $W_1$  with  $C_1 = \text{cok}(C_0 \rightarrow W_1)$  also by Theorem 3.7. The result holds by repeating this process. □

**Definition 4.2.** *Let  $M$  be an  $R$ -module. By the  $\mathcal{L}_n$ -injective dimension  $L_n \text{id}_R M$  of  $M$  is defined to be the smallest integer  $m \geq 0$  such that the sequence  $0 \rightarrow M \rightarrow W_0 \rightarrow W_1 \rightarrow \cdots \rightarrow W_{m-1} \rightarrow W_m \rightarrow 0$  in which each  $W_i$  is  $L_n$ -injective for  $0 \leq i \leq m$  is exact. If there is no such integer  $m$ , set  $L_n \text{id}_R M = \infty$ .*

**Example 4.3.** *Let  $M$  be an  $R$ -module.*

- (1)  $M$  is  $L_n$ -injective if and only if  $L_n \text{id}_R M = 0$ .
- (2)  $L_n \text{id}_R M \leq \text{id}_R M$ .
- (3) If  $m \leq n$ , then  $L_m \text{id}_R M \leq L_n \text{id}_R M$  since every  $L_n$ -injective module is  $L_m$ -injective.

**Theorem 4.4.** *Let  $m$  be a nonnegative integer. The following statements are equivalent for an  $R$ -module  $N$ :*

- (1)  $L_n \text{id}_R N \leq m$ .
- (2)  $\text{Ext}_R^{m+1}(M, N) = 0$  for any  $M \in \mathcal{F}_n$ .
- (3)  $\text{Ext}_R^{m+i}(M, N) = 0$  for any  $M \in \mathcal{F}_n$  and any  $i \geq 1$ .
- (4) If  $0 \rightarrow N \rightarrow W_0 \rightarrow W_1 \rightarrow \cdots \rightarrow W_{m-1} \rightarrow W_m \rightarrow 0$  is exact, where  $W_0, W_1, \dots, W_{m-1}$  are  $L_n$ -injective, then  $W_m$  is  $L_n$ -injective.
- (5) If  $0 \rightarrow N \rightarrow W_0 \rightarrow W_1 \rightarrow \cdots \rightarrow W_{m-1} \rightarrow W_m \rightarrow 0$  is exact, where  $W_0, W_1, \dots, W_{m-1}$  is injective, then  $W_m$  is  $L_n$ -injective.
- (6) The  $m$ -cosyzygy of  $N$  in its minimal  $L_n$ -injective resolution is  $L_n$ -injective.



**Proof.** (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6). Trivially.

(1) $\Rightarrow$ (2). Since  $L_n \text{id}_R N \leq m$ , there is an exact sequence  $0 \rightarrow N \rightarrow W_0 \rightarrow W_1 \rightarrow \cdots \rightarrow W_{m-1} \rightarrow W_m \rightarrow 0$  in which each  $W_i$  is  $L_n$ -injective. Therefore, for  $M \in \mathcal{F}_n$ ,  $\text{Ext}_R^{m+1}(M, N) \cong \text{Ext}_R^1(M, W_m) = 0$ .

(2) $\Rightarrow$ (3). For  $M \in \mathcal{F}_n$ , there is an exact sequence  $0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$  with  $P$  projective. Then  $\text{Ext}_R^k(M, N) \cong \text{Ext}_R^{k-1}(K, N)$ . Thus it follows by induction.

(6) $\Rightarrow$ (1). Take any minimal  $L_n$ -injective resolution of  $M : 0 \rightarrow M \rightarrow E_0 \rightarrow E_1 \rightarrow \cdots \rightarrow E_{m-1} \rightarrow L^{m-1} \rightarrow 0$  with each  $E_i$  be  $L_n$ -injective and  $L^{m-1}$  be the  $m$ -th cosyzygy of  $M$ . Then the result follows since  $L^{m-1}$  is  $L_n$ -injective by hypothesis.  $\square$

**Proposition 4.5.** *Let  $0 \rightarrow A \rightarrow W \rightarrow C \rightarrow 0$  be an exact sequence, where  $W$  is  $L_n$ -injective.*

- (1) *If  $A$  is  $L_n$ -injective, then so is  $C$ .*
- (2) *If  $L_n \text{id}_R A > 0$ , then  $L_n \text{id}_R C = L_n \text{id}_R A - 1$ .*

**Proof.** (1) It follows by Theorem 2.6.

- (2) It follows from the isomorphism  $\text{Ext}_R^m(M, C) \cong \text{Ext}_R^{m+1}(M, A)$ , where  $M \in \mathcal{F}_n$ .  $\square$

**Proposition 4.6.** *Let  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  be an exact sequence.*

- (1) *If two of  $L_n \text{id}_R A$ ,  $L_n \text{id}_R B$ , and  $L_n \text{id}_R C$  are finite, so is the third.*
- (2) *If  $A$  is  $L_n$ -injective, then  $L_n \text{id}_R B = L_n \text{id}_R C$ .*
- (3)  $L_n \text{id}_R B \leq \max\{L_n \text{id}_R A, L_n \text{id}_R C\}$ .
- (4)  $L_n \text{id}_R C \leq \max\{L_n \text{id}_R A - 1, L_n \text{id}_R B\}$ .
- (5)  $L_n \text{id}_R A \leq \max\{L_n \text{id}_R B, L_n \text{id}_R C + 1\}$ .

**Proof.** These results follow easily from the exact sequence

$$\text{Ext}_R^m(M, C) \rightarrow \text{Ext}_R^{m+1}(M, A) \rightarrow \text{Ext}_R^{m+1}(M, B) \rightarrow \text{Ext}_R^{m+1}(M, C) \rightarrow \text{Ext}_R^{m+2}(M, A)$$

and applying Theorem 4.4, where  $M \in \mathcal{F}_n$ .  $\square$

**Proposition 4.7.** *Let  $\{M_i \mid i \in \Gamma\}$  be a family of  $R$ -modules. Then*

$$L_n \text{id}_R \left( \prod_{i \in \Gamma} M_i \right) = \sup\{L_n \text{id}_R M_i \mid i \in \Gamma\}.$$

**Proof.** It is straightforward.  $\square$

**Theorem 4.8.** *Let  $N$  be an  $R$ -module. Then  $L_n \text{id}_R F_n(N) = L_n \text{id}_R N$ .*

**Proof.** This follows from the first exact sequence in Theorem 3.7 (3) and Proposition 4.6.  $\square$

**Proposition 4.9.** *Let  $N$  be an  $R$ -module. If  $\text{id}_R N < \infty$  and every injective  $R$ -module has the flat dimension at most  $n$ , then  $L_n \text{id}_R N = \text{id}_R N$ .*

**Proof.** Write  $\text{id}_R N = m$ . Then  $L_n \text{id}_R N \leq m$ . Pick an injective  $R$ -module  $E$  such that  $\text{Ext}_R^m(E, N) \neq 0$ . As  $E \in \mathcal{F}_n$  we get  $L_n \text{id}_R N \geq m$ . Hence  $L_n \text{id}_R N = m$ .  $\square$

**Theorem 4.10.** *Let  $N$  be an  $R$ -module with  $L_n \text{id}_R N = m < \infty$ . Then there exists an  $L_n$ -injective module  $W \in \mathcal{F}_n$  such that  $\text{Ext}_R^m(W, N) \neq 0$ .*

**Proof.** Since  $L_n \text{id}_R N = m$ , we pick an  $R$ -module  $M \in \mathcal{F}_n$  such that  $\text{Ext}_R^m(M, N) \neq 0$ . Let  $0 \rightarrow M \rightarrow W \rightarrow B \rightarrow 0$  be exact, where  $W$  is the  $L_n$ -injective envelope. By Theorem 3.7  $B \in \mathcal{F}_n$ . By Theorem 3.8,  $W \in \mathcal{F}_n$ . Since the sequence  $\text{Ext}_R^m(W, N) \rightarrow \text{Ext}_R^m(M, N) \rightarrow \text{Ext}_R^{m+1}(B, N) = 0$  is exact, we obtain  $\text{Ext}_R^m(W, N) \neq 0$ .  $\square$

## 5. $L_n$ -GLOBAL DIMENSIONS OF A RING

To characterize properties of rings by using  $L_n$ -injectivity, we are in the position to define the  $L_n$ -global dimension of a ring.

**Definition 5.1.** For a ring  $R$ , its left  $L_n$ -global dimension  $l.L_n \text{dim}(R)$  is defined by

$$l.L_n \text{dim}(R) = \sup\{L_n \text{id}_R M \mid M \text{ is an } R\text{-module}\}.$$

**Example 5.2.** For a ring  $R$ , we have:

- (1)  $l.L_n \text{dim}(R) \leq l.\text{gl.dim}(R)$ .
- (2) If  $m \leq n$ , then  $l.L_m \text{dim}(R) \leq l.L_n \text{dim}(R)$ .

**Example 5.3.** By [13, Lemma 3.6] and [8, Corollary 6.4], a domain  $R$  is APD if and only if  $L_1 \text{dim}(R) \leq 1$ .

**Theorem 5.4.** Let  $m$  be a nonnegative integer. Then the following statements are equivalent for a ring  $R$ :

- (1)  $l.L_n \text{dim}(R) \leq m$ .
- (2)  $\text{Ext}_R^{m+i}(M, N) = 0$  for any  $M \in \mathcal{F}_n$  and  $N \in {}_R \mathfrak{M}$  and for any  $i \geq 1$ .
- (3)  $\text{Ext}_R^{m+1}(M, N) = 0$  for any  $M \in \mathcal{F}_n$  and  $N \in {}_R \mathfrak{M}$ .
- (4)  $\text{Ext}_R^{m+i}(M, N) = 0$  for any  $M, N \in \mathcal{F}_n$  and any  $i \geq 1$ .
- (5)  $\text{Ext}_R^{m+1}(M, N) = 0$  for any  $M, N \in \mathcal{F}_n$ .
- (6)  $\sup\{L_n \text{id}_R N \mid N \in \mathcal{F}_n\} \leq m$ .
- (7)  $\sup\{\text{pd}_R M \mid M \in \mathcal{F}_n\} \leq m$ .

**Proof.** (2) $\Rightarrow$ (4) $\Rightarrow$ (5) and (3) $\Rightarrow$ (7) are trivial.

(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). It follows from Theorem 4.4.

(5) $\Rightarrow$ (6). Let  $N \in \mathcal{F}_n$ . Then  $L_n \text{id}_R N \leq m$  by applying Theorem 4.4 again. Therefore,  $\sup\{L_n \text{id}_R N \mid N \in \mathcal{F}_n\} \leq m$ .

(6) $\Rightarrow$ (1). Let  $N \in {}_R \mathfrak{M}$ . Then we have the exact sequence  $0 \rightarrow A \rightarrow F_n(N) \rightarrow N \rightarrow 0$  by Theorem 3.7, where  $A \in \mathcal{L}_n$ . By Proposition 4.6,  $L_n \text{id}_R N = L_n \text{id}_R F_n(N) \leq m$ . Therefore,  $l.L_n \text{dim}(R) \leq m$ .

(7) $\Rightarrow$ (3). Let  $M \in \mathcal{F}_n$  and  $N \in {}_R \mathfrak{M}$ . Since  $\text{pd}_R M \leq m$ , we have  $\text{Ext}_R^{m+1}(M, N) = 0$ .  $\square$

**Corollary 5.5.** For any ring  $R$ , the following are identical:

- (1)  $l.L_n \text{dim}(R)$ .
- (2)  $\sup\{L_n \text{id}_R N \mid N \in \mathcal{F}_n\}$ .
- (3)  $\sup\{\text{pd}_R M \mid M \in \mathcal{F}_n\}$ .

**Theorem 5.6.** Let  $m$  be a nonnegative integer. If  $l.L_n \text{dim}(R) < \infty$ , then the following statements are equivalent:

- (1)  $l.L_n\dim(R) \leq m$ .
- (2)  $\sup\{\text{pd}_R M \mid M \in \mathcal{F}_n \cap \mathcal{L}_n\} \leq m$ .
- (3)  $\sup\{\text{pd}_R L_n(M) \mid M \in \mathcal{F}_n\} \leq m$ .
- (4)  $\sup\{\text{pd}_R F_n(M) \mid M \in \mathcal{L}_n\} \leq m$ .
- (5)  $\sup\{\text{pd}_R F_n(M) \mid M \in {}_R\mathfrak{M}\} \leq m$ .
- (6)  $\sup\{L_n\text{id}_R M \mid M \text{ is projective}\} \leq m$ .

**Proof.** (1) $\Rightarrow$ (5). It follows from Theorem 5.4.

(5) $\Rightarrow$ (4). It is trivial.

(4) $\Rightarrow$ (2). For any  $M \in \mathcal{F}_n \cap \mathcal{L}_n$ ,  $F_n(M) = M$ , and hence  $\text{pd}_R M \leq m$  by hypothesis.

(2) $\Rightarrow$ (1). Let  $M \in \mathcal{F}_n$ . Because  $l.L_n\dim(R) < \infty$ , by Proposition 4.7 and Theorem 3.7 there is an exact sequence

$$0 \rightarrow M \rightarrow W_0 \rightarrow W_1 \rightarrow \cdots \rightarrow W_{k-1} \rightarrow W_k \rightarrow 0 \quad (5.1)$$

such that  $W_0, W_1, \dots, W_{k-1}$  are  $L_n$ -envelopes of some modules with the flat dimension at most  $n$ , and every  $\mathcal{L}_n$ -cosyzygy in (5.1) is in  $\mathcal{F}_n$ . Thus  $W_k \in \mathcal{F}_n \cap \mathcal{L}_n$ . By hypothesis and Theorem 3.8,  $\text{pd}_R W_i \leq m$  for  $i = 0, 1, \dots, k$ . These imply that  $\text{pd}_R M \leq m$ . Consequently,  $l.L_n\dim(R) \leq m$  by Theorem 5.4.

(2) $\Rightarrow$ (3). By Theorem 3.8,  $L_n(M) \in \mathcal{F}_n \cap \mathcal{L}_n$ .

(3) $\Rightarrow$ (2). For any  $M \in \mathcal{F}_n \cap \mathcal{L}_n$ ,  $L_n(M) = M$ , and hence  $\text{pd}_R M \leq m$  by hypothesis.

(1) $\Rightarrow$ (6). It is clear because every projective module is in  $\mathcal{F}_n$ .

(6) $\Rightarrow$ (1). Let  $N \in \mathcal{F}_n$ . Since  $l.L_n\dim(R) < \infty$ ,  $\text{pd}_R N < \infty$  by applying Theorem 5.4. Then we can pick a projective resolution of

$$0 \rightarrow P_k \rightarrow P_{k-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow N \rightarrow 0. \quad (5.2)$$

Because  $L_n\text{id}_R P_i \leq m$  by hypothesis for  $i = 0, 1, \dots, k$ ,  $L_n\text{id}_R N \leq m$  by applying repeatedly Proposition 4.6. So  $l.L_n\dim(R) \leq m$  by Theorem 5.4.  $\square$

**Corollary 5.7.** *Let  $n \leq m$ . The following statements are equivalent for any ring  $R$ :*

- (1)  $l.L_n\dim(R) \leq m$ .
- (2) *If  $\text{fd}_R M \leq n$ , then  $\text{pd}_R M \leq m$ .*
- (3) *If  $M$  is a submodule of a projective module  $P$  with  $M \in \mathcal{F}_{n-1}$ , then  $\text{pd}_R M \leq m - 1$ .*

**Proof.** It is easy by Theorem 5.4.  $\square$

**Corollary 5.8.**  $l.L_n\dim(R) \leq n$  if and only if  $\text{pd}_R M \leq n$  for any  $M \in \mathcal{F}_n$ .

**Proof.** It follows by taking  $m = n$  in Corollary 5.7.  $\square$

Recall that a ring  $R$  is called left  $m$ -perfect if the projective dimension of every flat module is at most  $m$  (see [7]). By the left global cotorsion dimension ( $l.\text{cot}.D(R)$ ) introduced by Mao and Ding [16] we have:

**Corollary 5.9.** [16, Corollary 19.27] *A ring  $R$  is left  $m$ -perfect if and only if  $l.\text{cot}.D(R) \leq m$ .*

**Proof.** It follows by taking  $n = 0$  in Corollary 5.7.  $\square$

**Theorem 5.10.** *Let  $m$  and  $k$  be nonnegative integers with  $n \leq k$  and let  $l.L_n\dim(R) \leq m$ . If  $M \in \mathcal{F}_k$ , then  $\text{pd}_R M \leq m + k - n$ .*

**Proof.** Let  $K$  be the  $(k - n)$ -th syzygy of  $M$  and let  $N$  be any  $R$ -module. Then  $\text{fd}_R(K) \leq n$ . Hence  $\text{Ext}_R^{m+k-n+1}(M, N) \cong \text{Ext}_R^{m+1}(K, N) = 0$  by Theorem 5.4. So  $\text{pd}_R(M) \leq m + k - n$ .  $\square$

**Corollary 5.11.** *Let  $k \geq n$ . If  $w.\text{gl.dim}(R) \leq k$  and  $l.L_n\dim(R) \leq m$ , then  $\text{gl.dim}(R) \leq m + k - n$ .*

## 6. THE CHARACTERIZATIONS OF RINGS

In this section we decide first when every  $L_n$ -injective module is injective.

**Theorem 6.1.** *The following statements are equivalent for a ring  $R$ :*

- (1)  $w.\text{gl.dim}(R) \leq n$ .
- (2) *Every  $L_n$ -injective module is injective.*
- (3) *If  $N \in \mathcal{L}_n$ , then  $\text{fd}_R N \leq n$ .*
- (4)  $\text{Ext}_R^1(M, W) = 0$  for any  $M, W \in \mathcal{L}_n$ .
- (5)  $\text{Ext}_R^i(M, W) = 0$  for any  $M, W \in \mathcal{L}_n$  and any  $i \geq 1$ .
- (6)  $\text{fd}_R L_n(M) \leq n$  for any  $M \in {}_R\mathfrak{M}$ .

**Proof.** (2) $\Rightarrow$ (5) $\Rightarrow$ (4) and (1) $\Rightarrow$ (6) are trivial.

(4) $\Rightarrow$ (3). By Theorem 2.8.

(6) $\Rightarrow$ (1). By Theorem 3.8,  $\text{fd}_R M \leq n$ . Hence  $w.\text{gl.dim}(R) \leq n$ .

(1) $\Rightarrow$ (2). Let  $M$  be any  $R$ -module and let  $W$  be an  $L_n$ -injective module. Then  $\text{fd}_R M \leq n$  by hypothesis. Hence we have  $\text{Ext}_R^1(M, W) = 0$ . Consequently,  $W$  is injective.

(2) $\Rightarrow$ (1). Let  $M$  be any  $R$ -module and let  $W$  be any  $L_n$ -injective module. Because  $W$  is injective by hypothesis,  $\text{Ext}_R^1(M, W) = 0$ . Therefore,  $\text{fd}_R M \leq n$  follows by Theorem 3.7, and hence  $w.\text{gl.dim}(R) \leq n$ .

(3) $\Rightarrow$ (2). Let  $W$  be an  $L_n$ -injective module and let  $0 \rightarrow W \rightarrow E \rightarrow C \rightarrow 0$  is an exact sequence, where  $E$  is injective. Then  $C$  is also  $L_n$ -injective. By hypothesis,  $\text{fd}_R C \leq n$ . Thus the exact sequence is split. Consequently,  $W$  is injective.  $\square$

**Theorem 6.2.** *Let  $w.\text{gl.dim}(R) < \infty$ . Then the following statements are equivalent:*

- (1)  $w.\text{gl.dim}(R) \leq n$ .
- (2)  $F_n(W)$  is injective for any  $W \in \mathcal{L}_n$ .
- (3) *If  $W \in \mathcal{L}_n \cap \mathcal{F}_n$ , then  $W$  is injective.*
- (4)  $L_n(N)$  is injective for any  $N \in \mathcal{F}_n$ .

**Proof.** (1) $\Rightarrow$ (2). By Theorem 3.8,  $F_n(W)$  is  $L_n$ -injective. By Theorem 6.1,  $F_n(W)$  is injective.

(2) $\Rightarrow$ (3). It is trivial as  $F_n(W) = W$ .

(3) $\Rightarrow$ (1). If there is an  $R$ -module with the flat dimension large than  $n$ , then there is an  $R$ -module  $M$  with  $\text{fd}_R M = n + 1$ . Take the exact  $0 \rightarrow A \rightarrow F_n(M) \rightarrow M \rightarrow 0$  as in (3.1). Then  $A$  is  $L_n$ -injective with  $\text{fd}_R A = n$ . Hence  $A$  is injective by hypothesis. Therefore, the given sequence is split. Hence  $\text{fd}_R M = n$ , a contradiction. Consequently,  $w.\text{gl.dim}(R) \leq n$ .

(3) $\Rightarrow$ (4). By Theorem 3.8,  $L_n(N) \in \mathcal{L}_n \cap \mathcal{F}_n$ . Hence  $L_n(N)$  is injective by hypothesis. If  $k > n$ ,  
 (4) $\Rightarrow$ (3). It is trivial as  $L_n(W) = W$ .  $\square$

**Corollary 6.3.** *The following statements are equivalent for a ring  $R$ :*

- (1)  $R$  is a Von Neumann Regular ring.
- (2) Every cotorsion module  $N$  is injective.
- (3) Every cotorsion module  $N$  is flat.
- (4)  $\text{Ext}_R^1(M, N) = 0$  for any cotorsion modules  $M, N$ .
- (5)  $\text{Ext}_R^i(M, N) = 0$  for any any cotorsion modules  $M, N$  and any  $i \geq 1$ .
- (6) The flat covers of any cotorsion module  $M$  are injective.

**Proof.** Take  $n = 0$  in Theorem 6.1.  $\square$

Let  $R$  be a ring. Bass defined in [1] the left finitistic projective dimension and left finitistic flat dimension of  $R$  as follow:

$$l.\text{FPD}(R) = \sup\{\text{pd}_R M \mid \text{pd}_R M < \infty\}.$$

$$l.\text{FFD}(R) = \sup\{\text{fd}_R M \mid \text{fd}_R M < \infty\}.$$

**Theorem 6.4.** *For any ring  $R$ ,  $l.L_n\text{dim}(R) \leq l.\text{FPD}(R)$ . Moreover, if  $m = l.\text{FPD}(R) < \infty$ , then  $l.\text{FPD}(R) = l.L_n\text{dim}(R)$  for any  $n > m$ .*

**Proof.** Let  $l.\text{FPD}(R) = k < \infty$  and let  $M \in \mathcal{F}_n$ . By [11, Proposition 6],  $\text{pd}_R M \leq k$ , and hence  $\text{Ext}_R^{k+1}(M, N) = 0$ . Therefore,  $l.L_n\text{dim}(R) \leq k$ .

If  $m = l.\text{FPD}(R) < \infty$ , there exist an  $R$ -module  $M$  with  $m = \text{pd}_R M < n$ . Then there exist an  $R$ -module  $N$  such that  $\text{Ext}_R^i(M, N) \neq 0$ . Hence  $L_n \text{id}_R N \geq m$  since  $M \in \mathcal{F}_n$ , and  $l.L_n\text{dim}(R) \geq m$ . That is,  $l.L_n\text{dim}(R) = l.\text{FPD}(R)$ .  $\square$

A ring  $R$  is called left perfect if every  $R$ -module has projective cover, equivalently. every flat  $R$ -module is projective. If  $R$  is commutative ring, then  $R$  is perfect if and only  $\text{FPD}(R) = 0$  (see [20]). By using the notion of  $L_n$ -injective modules, we can characterize left perfect rings.

**Theorem 6.5.** *Let  $n \geq 1$ . The following statements are equivalent for a ring  $R$ :*

- (1)  $l.L_n\text{dim}(R) = 0$ , that is, every module is  $L_n$ -injective.
- (2)  $M$  is projective for any  $M \in \mathcal{F}_n$ .
- (3)  $M$  is  $L_n$ -injective for any  $M \in \mathcal{F}_n$ .
- (4)  $\text{Ext}_R^1(M, N) = 0$  for any  $M, N \in \mathcal{F}_n$ .
- (5)  $\text{Ext}_R^i(M, N) = 0$  for any  $M, N \in \mathcal{F}_n$  and  $i \geq 1$ .
- (6)  $L_n(M)$  is projective for any  $M \in \mathcal{F}_n$ .
- (7)  $F_n(M)$  is projective for any  $M \in \mathcal{L}_n$ .
- (8)  $F_n(M)$  is  $L_n$ -injective for any  $M \in {}_R\mathfrak{M}$ .
- (9)  $R$  is left perfect and  $l.\text{FFD}(R) = 0$ .
- (10)  $l.\text{FPD}(R) = 0$ .

**Proof.** (1)  $\Leftrightarrow$  (2)  $\Leftrightarrow$  (3)  $\Leftrightarrow$  (4)  $\Leftrightarrow$  (5)  $\Leftrightarrow$  (6)  $\Leftrightarrow$  (7). By Theorem 5.4.

(1) $\Rightarrow$ (8) and (9)  $\Leftrightarrow$  (10). Trivial.

(8) $\Rightarrow$ (1). By Theorem 4.8.

(1) $\Rightarrow$ (9). Since every  $L_n$ -injective is cotorsion, we have that  $R$  is left perfect assertion is true by [21, Proposition 3.3.1]. By Corollary 2.10,  $l\text{-FFD}(R) = 0$ .

(9) $\Rightarrow$ (1). By Applying [21, Proposition 3.3.1] and Corollary 2.10 again.  $\square$

**Corollary 6.6.** *Let  $n \geq 1$ .*

(1) *If  $l\text{-FPD}(R) = 0$ , then every  $R$ -module is  $L_n$ -injective.*

(2) *A commutative  $R$  is perfect if and only if every  $R$ -module is  $L_n$ -injective.*

**Proof.** (1) It follows by Theorem 6.4.

(2) It is immediate from (1).  $\square$

**Definition 6.7.** *A ring  $R$  is called left  $L_n$ -hereditary if every quotient module of an  $L_n$ -injective module is  $L_n$ -injective. In other words,  $l.L_n\dim(R) \leq 1$ .*

**Example 6.8.** (1) *Left hereditary rings are certainly  $L_n$ -hereditary for all  $n \geq 0$ .*

(2) *Recall that a commutative ring  $R$  is called almost perfect if its proper epic images are perfect. An almost perfect domain is said simply an APD. In [18] it is shown that if  $R$  is almost perfect, then  $R$  is either perfect or APD. By [13, Lemma 3.6] and [8, Corollary 6.4], A domain  $R$  is  $L_1$ -hereditary if and only if  $R$  is an APD.*

(3) *Let  $R$  be a Noetherian domain. Then  $R$  is an APD if and only if  $\dim(R) \leq 1$  by [12, Theorem 90].*

(4) *Let  $R = R_1 \times \cdots \times R_n$ . Then  $R$  is  $L_n$ -hereditary if and only if each  $R_i$  is  $L_n$ -hereditary.*

**Theorem 6.9.** *Let  $n \geq 1$ . Then the following statements are equivalent for a ring  $R$ :*

(1)  *$R$  is  $L_n$ -hereditary.*

(2)  *$\text{pd}_R M \leq 1$  for any  $M \in \mathcal{F}_n$ .*

(3)  *$L_n\text{id}_R M \leq 1$  for any  $M \in \mathcal{F}_n$ .*

(4) *Every quotient module of an injective module is  $L_n$ -injective.*

(5)  *$E(M)/M$  is  $L_n$ -injective for any  $R$ -module  $M$ , where  $E(M)$  is an injective envelope of  $M$ .*

(6)  *$E(M)/M$  is  $L_n$ -injective for any  $R$ -module  $M$ , where  $E(M)$  is an  $L_n$ -injective envelope of  $M$ .*

(7) *Every submodule  $N \in \mathcal{F}_{n-1}$  of a projective  $R$ -module  $P$  is projective.*

**Proof.** (1)  $\Leftrightarrow$  (2)  $\Leftrightarrow$  (3). By Theorem 5.4.

(1) $\Rightarrow$ (4). It is trivial.

(4) $\Rightarrow$ (1). Let  $0 \rightarrow N \rightarrow W_0 \rightarrow W_1 \rightarrow 0$  be exact, where  $W_0$  is  $L_n$ -injective. Let  $E$  is the injective hull of  $W_0$  and write  $W = E/N$ . Then the following is a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \longrightarrow & W_0 & \longrightarrow & W_1 \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & N & \longrightarrow & E & \longrightarrow & W \longrightarrow 0 \end{array}$$

Therefore,  $0 \rightarrow W_0 \rightarrow E \oplus W_1 \rightarrow W \rightarrow 0$  is exact. By hypothesis,  $W$  is  $L_n$ -injective. By Proposition 2.5 and Example 2.3,  $W_1$  is  $L_n$ -injective.

(1) $\Rightarrow$ (6) $\Rightarrow$ (5). It is trivial.

(5) $\Rightarrow$ (4). Let  $0 \rightarrow K \rightarrow E \rightarrow C \rightarrow 0$  be exact, where  $E$  is injective. Set  $E(K) \subseteq E$  is the injective envelope of  $K$ . Then there exist an  $R$ -module  $E_0$  such that  $E = E(K) \oplus E_0$ . So  $C \cong E/K \cong (E(K) \oplus E_0)/K \cong (E(K)/K) \oplus E_0$ . Hence  $C$  is  $L_n$ -injective since  $E(K)/K$  is  $L_n$ -injective by hypothesis.

(4) $\Rightarrow$ (7). Let  $P$  be a projective  $R$ -module and  $N \in \mathcal{F}_{n-1}$  be a submodule of  $P$  and  $X$  be any  $R$ -module. Then there exist exact sequences  $0 \rightarrow N \rightarrow P \rightarrow P/N \rightarrow 0$  with  $P/N \in \mathcal{F}_n$  and  $0 \rightarrow X \rightarrow E \rightarrow C \rightarrow 0$  with  $E$  injective. By hypothesis,  $C$  is  $L_n$ -injective. For any  $\alpha \in \text{Hom}_R(N, C)$ , consider the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \xrightarrow{f} & P & \xrightarrow{g} & P/N \longrightarrow 0 \\ & & \searrow & & \downarrow \alpha & \searrow & \\ & & X & \xrightarrow{\theta f'} & E & \xrightarrow{\beta g'} & C \longrightarrow 0 \end{array}$$

Then there exist  $\beta \in \text{Hom}_R(P, C)$  such that  $\alpha = \beta f$  by Proposition 2.4. So there exist  $\gamma \in \text{Hom}_R(P, E)$  such that  $\gamma = g'\beta$  since  $P$  is projective. There exist  $\theta = \gamma f \in \text{Hom}_R(N, E)$  such that  $\alpha = g'\theta$ . Then  $\text{Hom}_R(N, E) \rightarrow \text{Hom}_R(N, C) \rightarrow 0$  is exact. Hence  $N$  is projective since  $\text{Ext}_R^1(N, X) = 0$ .

(7) $\Rightarrow$ (1). Let  $W$  be an  $L_n$ -injective  $R$ -module and  $N$  be its submodule and  $A \in \mathcal{F}_n$ . Then there exist an sequence  $0 \rightarrow K \rightarrow P \rightarrow A \rightarrow 0$  with  $P$  projective and  $K \in \mathcal{F}_{n-1}$ . By hypothesis,  $K$  is projective. For any  $\alpha \in \text{Hom}_R(K, W/N)$ , consider the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \xrightarrow{f} & P & \xrightarrow{g} & A \longrightarrow 0 \\ & & \searrow & & \downarrow \alpha & \searrow & \\ & & N & \xrightarrow{\beta f'} & W & \xrightarrow{g'} & W/N \longrightarrow 0 \end{array}$$

Then there exist  $\beta \in \text{Hom}_R(K, W)$  such that  $\alpha = g'\beta$ . By hypothesis,  $W$  is  $L_n$ -injective and  $A \in \mathcal{F}_n$ , then exist  $\gamma \in \text{Hom}_R(P, W)$  such that  $\beta = \gamma f$  by Proposition 2.4. Then  $\text{Hom}_R(P, W/N) \rightarrow \text{Hom}_R(K, W/N) \rightarrow 0$  is exact. Hence  $W/N$  is  $L_n$ -injective since  $\text{Ext}_R^1(A, W/N) = 0$ .  $\square$

**Corollary 6.10.** *Let  $n \geq 1$ . Let  $R$  be a commutative ring and let  $S$  be a multiplicatively closed subset of  $R$ . If  $R$  is an  $L_n$ -hereditary ring, then so is  $R_S$ .*

**Proof.** Let  $M$  be an  $R_S$ -module with  $\text{fd}_{R_S} M \leq n$ . Then  $\text{fd}_R M \leq n$ . By Theorem 6.9,  $\text{pd}_R M \leq 1$ . Certainly,  $\text{pd}_{R_S} M \leq 1$ . Consequently,  $R_S$  is  $L_n$ -hereditary.  $\square$

**Corollary 6.11.** *Then the following statements are equivalent for a ring  $R$ :*

- (1)  $l.L_1\dim(R) \leq 1$ .
- (2) Every quotient module of an injective module is  $L_1$ -injective.
- (3) Every flat submodule of a projective module is projective.
- (4) If  $\text{fd}_R M \leq 1$ , then  $\text{pd}_R M \leq 1$ .

**Theorem 6.12.** *Let  $n > 1$ . Then the following statements are equivalent for a ring  $R$ :*

- (1)  $l.L_n\dim(R) \leq 1$ .
- (2)  $l.L_2\dim(R) \leq 1$ .
- (3)  $l.\text{FPD}(R) \leq 1$ .

(4)  $l.L_1\dim(R) \leq 1$  and  $l.\text{FFD}(R) \leq 1$ .

**Proof.** (1)  $\Rightarrow$  (2). It is trivial.

(2) $\Rightarrow$ (3). Let  $M$  be an  $R$ -module with  $k := \text{pd}_R M < \infty$ . If  $k > 1$ , then it is certain that there is an  $R$ -module  $N$  with  $\text{pd}_R N = 2$ . By hypothesis  $\text{pd}_R N \leq 1$ , a contradiction. So we obtain  $k \leq 1$ , whence  $l.\text{FPD}(R) \leq 1$ .

(3) $\Rightarrow$ (1). If  $M \in \mathcal{F}_n$ , then  $\text{pd}_R M < \infty$  by [11, Proposition 6]. Thus  $\text{pd}_R M \leq 1$  by hypothesis. Hence  $l.L_n\dim(R) \leq 1$ .

(1)  $\Rightarrow$  (4). Certainly,  $l.L_1\dim(R) \leq l.L_n\dim(R) \leq 1$ . For any  $M \in \mathcal{F}_n$ , we have  $\text{fd}_R M \leq \text{pd}_R M \leq 1$  by Theorem 6.9. Hence  $\mathcal{F}_1 = \mathcal{F}_n$ . By Theorem 2.9,  $\text{FFD}(R) \leq 1$ .

(4) $\Rightarrow$ (1). Since  $l.\text{FFD}(R) \leq 1$ ,  $\mathcal{L}_1 = \mathcal{L}_n$  by using Theorem 2.9 again. Hence we have  $l.L_n\dim(R) = l.L_1\dim(R) \leq 1$ .  $\square$

By [11, Proposition 6] and [8, Corollary 6.4], we have the following corollary:

**Corollary 6.13.** *A domain  $R$  is an APD if and only if  $\text{FPD}(R) \leq 1$ .*

**Theorem 6.14.** *Let  $n \geq 1$ . Then the following statements are equivalent for a ring  $R$ :*

- (1)  $R$  is left hereditary.
- (2)  $R$  is  $L_n$ -hereditary and  $w.\text{gl.dim}(R) \leq 1$ .
- (3)  $R$  is  $L_1$ -hereditary and  $w.\text{gl.dim}(R) \leq 1$ .
- (4)  $R$  is  $L_n$ -hereditary and  $w.\text{gl.dim}(R) < \infty$ .
- (5)  $R$  is  $L_n$ -hereditary and  $w.\text{gl.dim}(R) \leq n$ .

**Proof.** (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3) and (2)  $\Rightarrow$  (4) are trival.

(3) $\Rightarrow$ (1). By Theorem 6.1, every  $L_1$ -injective module is injective. Hence  $R$  is hereditary.

(5) $\Rightarrow$ (1). Be similar to (3) $\Rightarrow$ (1).

(4)  $\Rightarrow$  (5). It is clear for the case  $n = 1$  by the argument above. Now we let  $n > 1$ . If  $k := w.\text{gl.dim}(R) > n$ , then there is an  $M \in \mathcal{F}_k$ . Let  $B$  be the  $(k - n)$ -syzygy of  $M$ , then  $\text{fd}_R B \leq n$ . By Theorem 6.9,  $\text{fd}_R B \leq \text{pd}_R B \leq 1$ . Hence  $k = \text{fd}_R M \leq k - n + 1$ . Thus  $n \leq 1$ , a contradiction. So  $k \leq n$ , which implies  $w.\text{gl.dim}(R) \leq n$ .  $\square$

**Theorem 6.15.** *Let  $R$  be a Noetherian domain with  $\dim(R) \leq 1$ . Then  $R$  is  $L_2$ -hereditary.*

**Proof.** By [10],  $\text{FPD}(R) = \dim(R) \leq 1$ . Hence  $R$  is  $L_2$ -hereditary by Theorem 6.12.  $\square$

## 7. EXAMPLES

Let  $M$  be an  $R$ -module. We say that  $M$  is torsion-free if  $ux = 0$  implies  $x = 0$ , where  $u$  is a non-zero-divisor of  $R$  and  $x \in M$ . It is well-known that a flat  $R$ -module is certainly torsion-free.

**Lemma 7.1.** *Let  $R$  be a commutative ring and let  $u \in R$  be neither a zero divisor nor a unit. Set  $\overline{R} = R/Ru$ . If  $A$  be a nonzero  $\overline{R}$ -module with  $\text{pd}_{\overline{R}} A < \infty$ , then  $\text{pd}_R A = \text{pd}_{\overline{R}} A + 1$ .*

**Proof.** Let  $k = \text{pd}_{\overline{R}} A$ . Because of  $\text{pd}_R \overline{R} = 1$ , we get  $\text{pd}_R A \leq \text{pd}_{\overline{R}} A + 1 = k + 1$  by Chang Theorem of rings. If  $k = 0$ , then we have  $\text{pd}_R A = 1$  since  $A$  is not torsion-free. Hence the assertion holds for  $k = 0$ .



Let  $k > 0$ . By [17, Exercise 9.6] there is a free  $R$ -module  $F$  such that  $\text{Ext}_R^k(A, F/aF) \neq 0$ . By Rees Theorem (see [17, Theorem 9.37], we have  $\text{Ext}_R^{k+1}(A, F) \neq 0$ . Hence  $\text{pd}_R A \geq k + 1$ . Thus we get  $\text{pd}_R A = k + 1$ .  $\square$

**Lemma 7.2.** *Let  $R$  be a domain and let  $J$  be an ideal of  $R$  generated by a regular sequence  $u_1, \dots, u_n$ . Then we have:*

- (1) *If  $M$  is a nonzero  $R/J$ -module with  $\text{pd}_{R/J} M < \infty$ , then  $\text{pd}_R M = \text{pd}_{R/J} M + n$ .*
- (2)  *$\text{pd}_R R/J = n$ .*
- (3) *For all  $k < n$ ,  $\text{Ext}_R^k(R/J, R) = 0$ .*
- (4) *For all  $R/J$ -modules  $M$  and any  $k < n$ ,  $\text{Ext}_R^k(M, R) = 0$ .*
- (5) *Let  $R$  be coherent and set  $T = \text{Ext}_R^n(R/J, R)$ . Then  $\text{Ext}_R^n(T, R) \cong R/J$ . Therefore,  $T \neq 0$ .*

*By the way,  $\text{pd}_R T = n$ .*

- (6) *If  $C$  is a  $(n-1)$ -cosyzygy of  $R$ , then  $\text{Ext}_R^1(R/J, L_{n-1}(C)) \neq 0$ .*

**Proof.** (1) Set  $R_1 = R/(u_1)$ . If  $n = 1$ , the assertion holds by Lemma 7.1. Now we assume  $n > 1$ . Then  $\bar{u}_2, \dots, \bar{u}_n$  is a regular sequence in  $R_1$ . Thus we may assume by induction that  $\text{pd}_{R_1} M = \text{pd}_{R/J} M + (n-1)$ . Using Lemma 7.1 again we get  $\text{pd}_R M = \text{pd}_{R_1} M + 1 = \text{pd}_{R/J} M + n$ .

- (2) This is direct from (1) by taking  $M = R/J$ .

(3) Since  $R/J$  is a torsion module,  $\text{Ext}_R^0(R/J, R) = \text{Hom}_R(R/J, R) = 0$  for  $n \geq 1$ . Let  $n > 1$ . Then  $\text{Ext}_R^1(R/J, R) = \text{Hom}_{R_1}(R/J, R_1) = 0$  by Rees Theorem. Therefore, the assertions for  $n = 1$  and  $n = 2$  hold. Let  $n > 2$  and assume by induction  $\text{Ext}_{R_1}^k(R/J, R_1) = 0$  for  $k < n-1$ . By Rees Theorem we obtain  $\text{Ext}_R^{k+1}(R/J, R) = \text{Ext}_{R_1}^k(R/J, R_1) = 0$ .

(4) Let  $0 \rightarrow A \rightarrow F \rightarrow M \rightarrow 0$  be an exact sequence, where  $F$  is a free  $R/J$ -module, that is,  $F = \bigoplus R/J$ . Thus we have  $\text{Ext}_R^k(F, R) = \prod \text{Ext}_R^k(R/J, R) = 0$  by (1). By induction we assume  $\text{Ext}_R^{k-1}(A, R) = 0$ . From the exact sequence  $\text{Ext}_R^{k-1}(A, R) \rightarrow \text{Ext}_R^k(M, R) = \text{Ext}_R^k(F, R) = 0$  we have  $\text{Ext}_R^k(M, R) = 0$ .

- (5) Let

$$0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_1 \rightarrow R \rightarrow R/J \rightarrow 0 \quad (7.1)$$

be a projective resolution of  $R/J$  in which each  $P_i$  is finitely generated. By taking the dual and using the facts  $\text{Ext}_R^k(R/J, R) = 0$  for  $k < n$  we obtain the following resolution

$$0 \rightarrow R^* \rightarrow P_1^* \rightarrow \dots \rightarrow P_{n-1}^* \rightarrow P_n^* \rightarrow T \rightarrow 0. \quad (7.2)$$

Note that  $T$  is an  $R/J$ -module. From (1) we have  $\text{Ext}_R^k(T, R) = 0$  for  $k < n$ . By double dual we have the following exact sequence

$$0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_1 \rightarrow R \rightarrow \text{Ext}_R^n(T, R) \rightarrow 0.$$

It follows that  $\text{Ext}_R^n(T, R) \cong R/J$ .

From the resolution (7.2) and the fact  $\text{Ext}_R^n(T, R) \cong R/J$  we obtain  $\text{pd}_R T = n$ .

(6) Note that  $\text{Ext}_R^1(R/J, C) \cong \text{Ext}_R^n(R/J, R) \neq 0$ . Consider the exact sequence  $0 \rightarrow C \rightarrow L_{n-1}(C) \rightarrow B \rightarrow 0$  in (3.1). Then we have the following exact sequence

$$\text{Ext}_R^n(R/J, L_n(C)) \rightarrow \text{Ext}_R^n(R/J, C) \rightarrow \text{Ext}_R^{n+1}(R/J, B) = 0.$$

Hence  $\text{Ext}_R^1(R/J, L_{n-1}(C)) \neq 0$ .  $\square$

**Example 7.3.** Now we exhibit a ring  $R$  in which  $\mathcal{L}_0 \supset \mathcal{L}_1 \supset \mathcal{L}_2 \supset \cdots \supset \mathcal{L}_n \supset \cdots$ . To do this, we take  $F$  be a field and set  $R = F[x_1, x_2, \dots, x_n, \dots]$ , where  $x_1, x_2, \dots, x_n, \dots$  are indeterminates over  $F$ . Then  $R$  is a coherent domain. For any  $n \geq 1$ , set  $J = (x_1, x_2, \dots, x_n)$ . Then  $\text{Ext}_R^n(R/J, R) \neq 0$  by Lemma 7.2. Let  $C_{n-1}$  be a  $(n-1)$ -cosyzygy of  $R$ . By using Lemma 7.2 again,  $L_{n-1}(C_{n-1})$  is an  $L_{n-1}$ -injective module but not  $L_n$ -injective. Thus we are done.

**Example 7.4.** There is a ring  $R$  that is  $L_1$ -hereditary but not almost perfect. In fact, let  $D$  be an APD but not a field. Then  $D$  is not perfect. Thus  $R = D \times D$  is an  $L_1$ -hereditary ring and  $I = (D, 0)$  is a nonzero ideal of  $R$ . Then  $D \cong R/I$  is a proper epic image of  $R$  but not perfect. Hence  $R$  is not almost perfect.

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